Therefore, as it could be intuitively stated, we see that a given subcarrier contributes better to the improvements on the SNR estimation accuracy (on another subcarrier) when it experiences higher SNR values.

Finally, we recall that the CRLB achieved on one of the remaining \( K - L \) subcarriers is exactly the same that could be achieved by any subcarrier when it is processed in a single-carrier system, hypothetically.

V. CONCLUSION

Analytical expressions for the Cramér–Rao lower bounds on the variance of unbiased subcarrier SNR estimators, in multicarrier systems, are derived when the signal is corrupted by additive white Gaussian noise. The most general case was considered where no a priori information is assumed about the dependence that may exist between the channel coefficients across the different subcarriers. As intuitively expected, it was shown that exploiting the mutual information between the subcarriers leads to substantial improvement on the achievable performance of the unbiased subcarrier SNR estimators.

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Central Limit Theorems for Wavelet Packet Decompositions of Stationary Random Processes

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Abstract—This paper provides central limit theorems for the wavelet packet decomposition of stationary band-limited random processes. The asymptotic analysis is performed for the sequences of the wavelet packet coefficients returned at the nodes of any given path of the \( M \)-band wavelet packet decomposition tree. It is shown that if the input process is strictly stationary, these sequences converge in distribution to white Gaussian processes when the resolution level increases, provided that the decomposition filters satisfy a suitable property of regularity. For any given path, the variance of the limit white Gaussian process directly relates to the value of the input process power spectral density at a specific frequency.

Index Terms—Band-limited stochastic processes, spectral analysis, wavelet transforms.

I. INTRODUCTION

This paper addresses the statistical properties of the \( M \)-band discrete wavelet packet transform (\( M \)-DWPT). Specifically, an asymptotic analysis is given for the correlation structure and the distribution of the \( M \)-band wavelet packet coefficients of stationary random processes.

In [1] and [2], such a study is carried out without analyzing the role played by the path followed in the \( M \)-DWPT tree and that of the wavelet decomposition filters. In contrast, this paper emphasizes that, given a path of the \( M \)-DWPT, the sequence of the \( M \)-band wavelet packet coefficients obtained at resolution \( j \) in this path converges, in a distributional sense specified below, to a discrete white Gaussian
II. Preliminary Results

A. Tree Decomposition and Path Representations

Throughout $M$ is a natural number larger than or equal to two, and $j$ and $n$ always refer to nonnegative integers. As usual, $\mathbb{N} = \{1, 2, \ldots\}$ stands for the set of natural numbers and $\mathcal{Z}$ for the set of integers. The tree $T$ considered below is constructed as follows: $T$ has a root (or starting) node $U \equiv W_{0,0}$ and double indexed children nodes $W_{j,n}$, where $j \geq 1$ and $n \in \{0, 1, \ldots, M^j - 1\}$ for every fixed $j$. In this tree decomposition, the children of $W_{j,n}$ are defined to be $W_{j+1,m_t+m}$, where $m = 0, 1, \ldots, M - 1$. The index $j$ will be referred as the decomposition level and the index $n$ as the frequency index. We use the notation $T = \left( U, \{ W_{j,n} \}_{j \geq 1, n \in \{0,1,\ldots,M^j-1\}} \right)$. In what follows, a path is any sequence of spaces $(U, \{ W_{j,n} \}_{j \geq 1})$ such that $W_{j,n}$ is a child of $W_{j-1,n_{j-1}}$ for every $j \geq 1$, with $n_0 = 0$ by convention. Let $\mathcal{P}$ be a given path of $T$. This path is described by a sequence of nodes (spaces) where the frequency index is

$$n_j = M n_{j-1} + m_j$$

for $j \geq 1$, with $m_j \in \{0, 1, \ldots, M - 1\}$. Therefore, in path $\mathcal{P}$ and at each decomposition level $j$, the frequency index is $n_j = \sum_{k=1}^{j} m_k M^{j-k}$ in $\{0, 1, \ldots, M^j - 1\}$. By construction, path $\mathcal{P}$ can be associated with a unique $M$-ary sequence $(m_k)_{k \in \mathbb{N}}$ of elements of $\{0, 1, \ldots, M-1\}$. On the other hand, any frequency index $n \in \{0, 1, \ldots, M^j - 1\}$ at decomposition level $j \geq 1$ can be associated with a unique finite subsequence $(m_1, m_2, \ldots, m_j)$ of elements of $\{0, 1, \ldots, M-1\}$ such that $n = \sum_{k=1}^{j} m_k M^{j-k}$. This unique subsequence will hereafter be called the $M$-ary subsequence associated with the pair $(j, n)$. With the terminology and notation introduced above, the sole sequence of nodes $(j, n_j)_{j \geq 1}$ that specifies path $\mathcal{P}$ is such that the $M$-ary subsequence associated with $(j, n_j)$ results from the concatenation of the $M$-ary subsequence associated with $(j-1, n_{j-1})$ with the unique value $m_j$ such that (1) holds true.

In what follows, $T$ is an $M$-DWPT tree whose nodes are the orthogonal nested functional subspaces generated from a root space $U \subseteq L^2(\mathbb{R})$ by using wavelet paraunitary filters with impulse responses $h_m, m = 0, 1, 2, \ldots, M - 1$. For further details about the computation and the properties of $M$-DWPT filters, the reader is asked to refer to [3]. The Fourier transform of the paraunitary filter with impulse response $h_m, m = 0, 1, 2, \ldots, M - 1$, is hereafter defined by

$$H_m(\omega) = \frac{1}{\sqrt{M}} \sum_{t \in \mathbb{Z}} h_m[t] \exp(-i t \omega).$$

B. General Formulas on the $M$-DWPT

Let $\Phi$ be a function such that $\{ \tau_k \Phi : k \in \mathbb{Z} \}$ is an orthonormal system of $L^2(\mathbb{R})$, where $\tau_k \Phi : t \mapsto \Phi(t-k)$. Let $U$ be the closure of the space spanned by this orthonormal system. With notation similar to [4] and [5], the $M$-DWPT decomposition of the function space $U$ involves splitting $U$ into $M$ orthogonal subspaces (an easy extension [6, Lemma 10.5.1] established for the standard DWPT) so that

$$U = \bigoplus_{m=0}^{M-1} W_{1,m},$$

and recursively applying the following splitting:

$$W_{j,n} = \bigoplus_{m=0}^{M-1} W_{j+1,m_n+m}$$

for every natural number $j$ and every $n = 0, 1, 2, \ldots, M^j - 1$.

Given $j \geq 0$ and $n \in \{0, 1, \ldots, M^j - 1\}$, let us consider the wavelet packet space $W_{j,n}$ located at node $(j, n)$ of the wavelet packet tree. This function space is the closure of the space spanned by the orthonormal set of the wavelet packet functions $\{\psi_{j,n,k} : k \in \mathbb{Z}\}$, with

$$\psi_{j,n,k}(t) = \psi_{j,n}(t - M^j k)$$

where the sequence $\{\psi_{j,n,k}\}_{j,n,k}$ is recursively defined by putting $\psi_{0,0} = \Phi$ and setting, for any $p \geq 0$, any $q \geq 0$ and any $m \in \{0, 1, \ldots, M - 1\}$

$$\psi_{p+1, M^p m + n}(t) = \sum_{t \in \mathbb{Z}} h_m[t] \psi_{p, n}(t - M^p t).$$

Each $\psi_{j,n}, j \geq 1$ and $n \in \{0, 1, \ldots, M^j - 1\}$, is thus obtained from $\Phi$ and the particular sequence of filters $(h_{m_1}, h_{m_2}, \ldots, h_{m_j})$, where $(m_1, m_2, \ldots, m_j)$ is the $M$-ary subsequence associated with $(j, n)$.

The standard formulas [4, p. 324, (8.10) and (8.11)] are obtained by setting $M = 2$ above. It is worth emphasizing that $\Phi$ is not necessarily the scaling function associated with the low-pass filter $h_0$. In other words, the $M$-DWPT applies to the general case where the input decomposition space $U$ is not necessarily the space generated by the translated versions of the scaling function associated with $h_0$.

We thus can fix an input space and decompose it by using different types of wavelet paraunitary filters. This is exactly what is done in Sections III and IV, where the input functional space is always the standard Paley–Wiener space but different $M$-DWPT filters are used to decompose it. In the particular case where $\Phi$ is the scaling function associated with $h_0$, the standard scaling equation [4, p. 228, (7.28)] implies that $\psi_{1,0}(t) = (1/\sqrt{2})\Phi(t/2)$.

Given $f \in L^1(\mathbb{R}) \cup L^2(\mathbb{R})$, let $\mathcal{F} f$ henceforth stand for the Fourier transform of $f$, where $\mathcal{F}$ is given by

$$\mathcal{F} f(\omega) = \int_{\mathbb{R}} f(t) \exp(-i \omega t) dt$$

if $f \in L^1(\mathbb{R})$. Given any $j \geq 1$ and any $n \in \{0, 1, \ldots, M^j - 1\}$, a straightforward recurrence based on the Fourier transform of (4) leads to

$$\mathcal{F} \psi_{j,n}(\omega) = M^{j/2} \left[ \prod_{k=1}^{j} H_m(M^j \omega) \right] \mathcal{F} \Phi(\omega)$$

where $(m_1, m_2, \ldots, m_j)$ is the $M$-ary subsequence associated with $(j, n)$. This standard result will prove useful in the sequel.

C. Shannon $M$-DWPT and the Paley–Wiener Space of Band-Limited Functions

The Shannon $M$-DWPT filters are hereafter denoted $h^S_m$, for $m = 0, 1, \ldots, M - 1$. These filters are ideal low-pass, bandpass, and high-pass filters. We have

$$H^S_m(\omega) = \sum_{t \in \mathbb{Z}} \mathbb{1}_{\Delta_m}(\omega - 2\pi t)$$

where $\mathbb{1}_K$ denotes the indicator function of a given set $K$; $\mathbb{1}_K(x) = 1$ if $x \in K$ and $\mathbb{1}_K(x) = 0$, otherwise, and

$$\Delta_m = \left[ -\frac{(m+1)\pi}{M}, -\frac{m\pi}{M} \right] \cup \left[ \frac{m\pi}{M}, \frac{(m+1)\pi}{M} \right].$$
The scaling function $\Phi^S$ associated with these filters is defined for every $t \in \mathbb{R}$ by $\Phi^S(t) = \text{sinc}(t) = \sin(\pi t) / \pi t$ with $\Phi^S(0) = 1$. The Fourier transform of this scaling function is

$$\mathcal{F}\Phi^S = \mathbb{1}_{[-\pi, \pi]}.$$ (7)

The closure $\mathcal{U}^S$ of the space spanned by the orthonormal system $\{\varphi_k, \Phi^S : k \in \mathbb{Z}\}$ is then the Paley–Wiener (PW) space of those elements of $L^2(\mathbb{R})$ that are $\pi$ band-limited in the sense that their Fourier transform is supported within $[-\pi, \pi]$.

Let $X$ be any band-limited wide-sense stationary (WSS) random process whose spectrum is supported within $[-\pi, \pi]$. We have (see [7, Appendix D])

$$X[k] = \int_{\mathbb{R}} X(t) \Phi^S(t - k) \, dt$$ (8)

so that $\mathcal{U}^S$ is the natural representation space of such a process. Any $M$-DWPT of $X$ can thus be initialized with the samples $X[k], k \in \mathbb{Z}$.

Now, let us consider the Shannon $M$-DWPT of the PW space $\mathcal{U}^S$. The wavelet packet functions $\psi^S_{j,n}$ of this $M$-DWPT can be computed by means of (4) with $\Phi = \Phi^S$ and $h^S = h^S_m, m = 0, 1, \ldots, M - 1$. The Fourier transforms of these wavelet packet functions are given by Proposition I below, which extends [4, p. 328, Prop. 8.2] since the latter follows from the former with $M = 2$.

**Proposition 1:** For $j \geq 0$ and $n \in \{0, 1, \ldots, M^j - 1\}$, we have

$$\mathcal{F}\psi^S_{j,n} = \mathbb{1}_{(\Delta_{j,n} \cap \mathbb{N})}$$ (9)

where, for any nonnegative integer $k$

$$\Delta_{j,n} = \left\{ -\frac{(k + 1)\pi}{M^j}, -\frac{k\pi}{M^j} \right\} \cup \left\{ \frac{k\pi}{M^j}, \frac{(k + 1)\pi}{M^j} \right\}$$ (10)

and $G$ is the map defined by $G(0) = 0$ and recursively setting, for $m = 0, 1, \ldots, M - 1$ and $\ell = 0, 1, 2, \ldots$

$$G(M^j + m) = \begin{cases} MG(\ell) + m, & \text{if } G(\ell) \text{ is even} \\ MG(\ell) - m + M - 1, & \text{if } G(\ell) \text{ is odd} \end{cases}.$$ (11)

**Proof:** A routine exercise based on (6) and (7) and the recursive definition of the wavelet packet functions.

In the rest of the paper, we set, for any pair $(j, k)$ of nonnegative integers

$$\Delta^+_{j,n} = \left\{ \frac{k\pi}{M^j}, \frac{(k + 1)\pi}{M^j} \right\}.$$ (12)

### III. ASYMPTOTIC ANALYSIS FOR THE AUTOCORRELATION FUNCTIONS OF THE M-DWPT OF SECOND-ORDER WSS RANDOM PROCESSES

Let $X$ denote a zero-mean second-order real random process assumed to be continuous in quadratic mean. The autocorrelation function of $X$, denoted by $R$, is defined by $R(t, s) = \mathbb{E}[X(t)X(s)]$. Given $j \geq 1$ and $n \in \{0, 1, \ldots, M^j - 1\}$, the projection of $X$ on the wavelet packet space $\mathcal{W}^S$, yields a sequence of random variables, the wavelet packet coefficients of $X$

$$c_{j,n}[k] = \int_{\mathbb{R}} X(t)\psi_{j,n,k}(t) \, dt, \quad k \in \mathbb{Z}$$ (13)

provided that $\int_{\mathbb{R}} \int_{\mathbb{R}^2} R(t, s)\psi_{j,n,k}(t)\psi_{j,n,k}(s) \, dt \, ds < \infty$, which will be assumed in the rest of this paper since commonly used wavelet functions are compactly supported or have sufficiently fast decay. The sequence given by (13) defines the discrete random process $c_{j,n} = (c_{j,n}[k])_{k \in \mathbb{Z}}$ of the wavelet packet coefficients of $X$ at resolution level $j$ and for frequency index $n$.

#### A. Problem Formulation

Let $R_{j,n}$ stand for the autocorrelation function of the random process $c_{j,n}$. We have

$$R_{j,n}[k, \ell] = \mathbb{E}[c_{j,n}[k]c_{j,n}[\ell]] = \int_{\mathbb{R}^2} R(t, s)\psi_{j,n,k}(t)\psi_{j,n,\ell}(s) \, dt \, ds.$$ (14)

If $X$ is WSS, we write $R(t, s) = R(t - s)$ with some usual and slight abuse of language. From (14), it follows that

$$R_{j,n}[k, \ell] = \int_{\mathbb{R}^2} R(t)\psi_{j,n,k}(t + s)\psi_{j,n,\ell}(s) \, dt \, ds.$$ (15)

In the sequel, the spectrum $\gamma$ of $X$, that is, the Fourier transform of $R$, is assumed to exist. By taking into account that the Fourier transform of $\psi_{j,n,k}$ is

$$\mathcal{F}\psi_{j,n,k}(\omega) = \exp(-iM^j k\omega)\mathcal{F}\psi_{j,n}(\omega)$$

by using Fubini’s theorem and Parseval’s equality, we derive from (15) that $c_{j,n}$ is WSS. For any $k, \ell \in \mathbb{Z}$, and with the same abuse of language as above, the value $R_{j,n}[k, \ell]$ of the autocorrelation function of the discrete random process $c_{j,n}$ is $R_{j,n}[k - \ell]$ with

$$R_{j,n}[k] = \frac{1}{2\pi} \int_{\mathbb{R}} \gamma(\omega)|\mathcal{F}\psi_{j,n}(\omega)|^2 \exp(iM^j k\omega) \, d\omega.$$ (16)

The purpose of the next section is then to analyze the behavior of $R_{j,n}$ when $j$ tends to $\infty$, in the case of the Shannon filters and some families of filters that converge to the Shannon filters. From now on, the input decomposition space is assumed to be the PW space $\mathcal{U}^S$.

#### B. Asymptotic Decorrelation

Consider the Shannon $M$-DWPT, that is, the decomposition of $\mathcal{U}^S$ associated with the Shannon $M$-DWPT filters $(h^S_m)_{m=0,1,\ldots,M-1}$. With the same notation and terminology as in Section II, let $(m_\ell)_{\ell \in \mathbb{N}}$ be an $M$-ary sequence of elements of $\{0, 1, \ldots, M^j - 1\}$ and $\mathcal{P} = \{\mathcal{U}^S, (\mathcal{W}^S_{j,n,\ell})_{\ell \in \mathbb{N}}\}$ be the path associated with this subsequence in the Shannon $M$-DWPT decomposition tree. It follows from Proposition 1 that the support of $\mathcal{F}\psi^S_{j,n,\ell}$ is $\Delta_{j,G(n_{\ell})}$. For $j \in \mathbb{N}$, the sets $\Delta^+_{j,G(n_{\ell})}$ are nested closed intervals whose diameters tend to zero. Therefore, their intersection contains only one point henceforth denoted by $\omega_{\mathcal{P}}$. It then follows from (12) that

$$\omega_{\mathcal{P}} = \lim_{j \to \infty} \frac{G(n_{\ell})\pi}{M^j}.$$ (17)

Let $X$ be some zero-mean second-order WSS random process, continuous in quadratic mean, with spectrum $\gamma$. The autocorrelation function $R^S_{j,n,\ell}$ of the projection of $X$ on $\mathcal{W}^S_{j,n,\ell}$ derives from (16) and is given by

$$R^S_{j,n,\ell}[k] = \frac{1}{2\pi} \int_{\mathbb{R}} \gamma(\omega)|\mathcal{F}\psi^S_{j,n,\ell}(\omega)|^2 \exp(iM^j k\omega) \, d\omega.$$ (18)

From (9) and (18) and by taking into account that $\gamma$ is even, as the Fourier transform of the even function $R$, it follows that

$$R^S_{j,n,\ell}[k] = \frac{M^j}{\pi} \sum_{\gamma \in \mathcal{P}_{j,n,\ell}} \gamma(\omega) \cos(M^j k\omega) \, d\omega.$$ (19)
where $\Delta^+_\nu, c_{\nu}(n)$ is given by (12). When $X$ satisfies some additional assumptions, the following Theorem 1 states that the Shannon $M$-DWPT of $X$ yields coefficients that tend to be decorrelated when $j$ tends to infinity. One of these additional assumptions is that $X$ is band-limited in the sense that its spectrum is supported within $[-\pi, \pi]$.

**Theorem 1:** Let $X$ be a zero-mean second-order WSS random process, continuous in quadratic mean. Assume that the spectrum $\gamma$ of $X$ is an element of $L^\infty(\mathbb{R})$ and is supported within $[-\pi, \pi]$. Let $\mathcal{P} = (U^S_{j,n}, \{W^S_{j,n}\}_{j,n})$ be a Shannon $M$-DWPT decomposition path.

If the spectrum $\gamma$ of $X$ is continuous at point $\omega_P$, then

$$
\lim_{j \to \infty} R^S_{j,n}[k] = \gamma(\omega_P) \delta[k]
$$

(20)

uniformly in $k \in \mathbb{Z}$, where $R^S_{j,n}[k]$ is the autocorrelation function of the coefficients resulting from the projection of $X$ on $\mathcal{P}_{j,n}$.

**Proof:** The proof is an easy generalization of that of [7, Proposition 1], which concerns the standard wavelet packet transform ($M = 2$).

The foregoing theorem is mainly of theoretical interest since the Shannon $M$-DWPT filters have infinite supports and are not really suitable for practical purpose. In order to obtain a result of the same type for filters of practical interest, the $M$-DWPT of $U^S$ is now assumed to be performed by using decomposition filters of order $r, \tilde{h}^{[m]}_n, m = 0, 1, \ldots, M - 1$, whose Fourier transforms $\tilde{H}^{[m]}_n$ are defined by (2) and such that

$$
\lim_{j \to \infty} \tilde{H}^{[m]}_n = \tilde{H}^{[m]} (a.e.).
$$

(21)

Similarly to above, $W^S_{j,n} \subset U^S$ henceforth stands for the wavelet packet space at node $(j, n)$. The wavelet packet functions $\psi^S_{j,n}[k]$ of the $M$-DWPT under consideration are now calculated by applying (4) with $\Phi = \Phi^S$ and $h_0 = \tilde{h}^{[m]}_n, m = 0, 1, \ldots, M - 1$.

According to [8], the Daubechies filters satisfy (21) for $M = 2$ when $r$ is the number of vanishing moments of the Daubechies wavelet function. According to [9], the Battle–Lemarié filters also satisfy (21) for $M = 2$ when $r$ is the spline order of the Battle–Lemarié scaling function. The existence of such families for $M > 2$ remains an open issue to address in forthcoming work. However, it can reasonably be expected that general $M$-DWPT filters of the Daubechies or Battle–Lemarié type converge to the Shannon filters in the sense given above.

**Theorem 2:** Let $X$ be a zero-mean second-order WSS random process, continuous in quadratic mean. Assume that the spectrum $\gamma$ of $X$ is an element of $L^\infty(\mathbb{R})$ and is supported within $[-\pi, \pi]$. Assume that the $M$-DWPT of the PW space $U^S$ is achieved by using decomposition filters $h^{[m]}_n, m = 0, 1, \ldots, M - 1$, satisfying (21). Let $R^S_{j,n}$ stand for the autocorrelation function of the wavelet packet coefficients of $X$ with respect to the wavelet packet space $W^S_{j,n}$. We have

$$
\lim_{j \to \infty} R^S_{j,n}[k] = R^S_{j,n}[k]
$$

(22)

uniformly in $k \in \mathbb{Z}$, where $R^S_{j,n}$ is given by (19).

**Remark 1:** Albeit straightforward, the following equalities are useful in the sequel. Let $(m_1, m_2, \ldots, m_j)$ be the $M$-ary subsequence associated with a given pair $(j, n)$. From (5), the functions $\psi^S_{j,n}[k]$ satisfy (4) for the Shannon $M$-DWPT of $U^S$ such that

$$
\mathcal{F}\psi^S_{j,n}[\omega] = M^{j/2} \prod_{j=1}^j H^S_{j,m}(\omega^{M-1}) \mathcal{F}\Phi^S(\omega).
$$

(23)

In the same way, the functions $\psi^S_{j,n}[k]$, involved in (4) for the decomposition of $U^S$ via the filters $h^{[m]}_n$ introduced above, satisfy the following equality:

$$
\mathcal{F}\psi^S_{j,n}[\omega] = M^{j/2} \prod_{j=1}^j H^S_{j,m}(\omega^{M-1}) \mathcal{F}\Phi^S(\omega).
$$

(24)

In addition, from (21), (23), and (24), we have that

$$
\lim_{r \to \infty} \mathcal{F}\psi^S_{j,n}[\omega] = \mathcal{F}\psi^S_{j,n}(a.e.).
$$

(25)

**Proof (of Theorem 2):** The autocorrelation function $R^S_{j,n}[k]$ is given by (16) and is equal to

$$
R^S_{j,n}[k] = \frac{1}{2\pi} \int_{\mathbb{R}} \gamma(\omega) |\mathcal{F}\psi^S_{j,n}(\omega)|^2 \exp(i M^j k \omega) \mathrm{d}\omega.
$$

(26)

In addition, we have

$$
|\mathcal{F}\psi^S_{j,n}(\omega)|^2 \leq \frac{1}{M^j} \int_{\mathbb{R}} \gamma(\omega) \mathcal{F}\psi^S_{j,n}(\omega) \mathrm{d}\omega.
$$

(27)

From (23) and (24) and by taking into account that $|H^S_{j,m}(\omega)|$ and $|H^S_{j,m}(\omega)|$ are less than or equal to one (due to the paraunitarity of the $M$-DWPT filters), we obtain

$$
\mathcal{F}\psi^S_{j,n}(\omega) \leq M^j \mathcal{F}\Phi^S(\omega).
$$

(28)

The result then derives from (25), (27), (28), and Lebesgue’s dominated convergence theorem.

**IV. CENTRAL LIMIT THEOREMS**

In this section, we consider a zero-mean real random process $X$ that has finite cumulants and polyspectra. In what follows, $q$ is a natural number. Let us denote by

$$
\text{cum}(t, s_1, s_2, \ldots, s_q) = \text{cum}(X(t), X(s_1), X(s_2), \ldots, X(s_q))
$$

the cumulant of order $q+1$ of $X$. The above cumulant is hereafter assumed to belong to $L^2(\mathbb{R}^{q+1})$ and to be finite (see [10, Prop. 1]) for a discussion about the existence of this cumulant). With the notation introduced so far, the cumulant of order $q+1$ of the random process $c_{j,n}$ has the integral form given by

$$
\text{cum}_{c_{j,n}}[k, k_1, k_2, \ldots, k_q] = \text{cum}(c_{j,n}[k], c_{j,n}[k_1], c_{j,n}[k_2], \ldots, c_{j,n}[k_q])
$$

$$
= \int_{\mathbb{R}^{q+1}} \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_q \text{cum}(t, s_1, s_2, \ldots, s_q) \psi^S_{j,n}(t)
$$

$$
\times \psi^S_{j,n}(t_1(x_1)) \psi^S_{j,n}(t_2(x_2)) \cdots \psi^S_{j,n}(t_q(x_q)).
$$

(29)

If $X$ is assumed to be strictly stationary so that $\text{cum}(t, t + t_1, t + t_2, \ldots, t + t_q) = \text{cum}(t_1, t_2, \ldots, t_q)$, then $c_{j,n}$ is a strictly stationary random process with cumulants $\text{cum}_{c_{j,n}}[k, k_1, k_2, \ldots, k_q] = \text{cum}_{c_{j,n}}[k_1, k_2, \ldots, k_q]$. Assume also that $X$ has a polyspectrum $\gamma_q(\omega_1, \omega_2, \ldots, \omega_q) \in L^\infty(\mathbb{R}^q)$ for every natural number $q$ and every $(\omega_1, \omega_2, \ldots, \omega_q) \in \mathbb{R}^q$. The polyspectrum is the Fourier transform of the cumulant $c_{j,n}(t, t_1, \ldots, t_q)$. When $q = 1, \gamma_1$ is the spectrum of
X and is simply denoted γ as in Section III. Then, after some routine algebra, (29) reduces to
\[
\text{cum}_{\gamma,n}(k_1, k_2, \ldots, k_N) = \frac{1}{(2\pi)^n} \int \prod_{j=1}^{n} \text{d}W_{\omega_j} \exp \left( -i M^{\gamma}(k_1 \omega_1 + k_2 \omega_2 + \cdots + k_N \omega_N) \right) \times \gamma(\omega_1, \omega_2, \ldots, \omega_N) \times \mathcal{F}^{(\gamma)}_{\psi,n}(\omega_1 - \omega_2, \ldots, -\omega_N) \times \mathcal{F}^{(\gamma)}_{\psi,n}(\omega_1) \cdots \mathcal{F}^{(\gamma)}_{\psi,n}(\omega_N).
\]

We then have the following.

**Theorem 3:** Let X be a zero-mean second-order strictly stationary random process, continuous in quadratic mean. Assume that the polyspectrum γ of X is an element of \(L^\infty(\mathbb{R}^n)\) for every natural number \(q \geq 1\) and that the spectrum γ of X is supported within \([-\pi, \pi]\). Consider the Shannon M-DWPT of the PSW process \(\mathcal{S}_X\). Let \(n \geq 2\), the cumulant of order \(q+1\), \(\text{cum}_{\gamma,n}(k_1, k_2, \ldots, k_N)\), of the discrete random process resulting from the projection of X on \(\mathcal{S}_X\). We have, uniformly in \(k_1, k_2, \ldots, k_N\),
\[
\lim_{n \to +\infty} \text{cum}_{\gamma,n}(k_1, k_2, \ldots, k_N) = 0
\]
uniquely in \(k_1, k_2, \ldots, k_N\).

**Proof:** It follows from (9) and (30) that the wavelet packet functions are the functions \(\psi_X^{(\gamma)}(\omega)\), the cumulant \(\text{cum}_{\gamma,n}(k_1, k_2, \ldots, k_N)\) of the discrete random process returned at node \((j, n)\) by the Shannon M-DWPT of X satisfies
\[
\text{cum}_{\gamma,n}(k_1, k_2, \ldots, k_N) \leq \frac{M^{(q+1)/2} \gamma}{(2\pi)^q} \int \Delta_{\psi,n}(\omega) d\omega_1 \cdots d\omega_q (32)
\]
where
\[
\Delta_{\psi,n}(\omega) = \Delta_{\psi,n}(\omega_1) \times \Delta_{\psi,n}(\omega_2) \times \cdots \times \Delta_{\psi,n}(\omega_N).
\]

Let \(n \to +\infty\),
\[
\lim_{n \to +\infty} \text{cum}_{\gamma,n}(k_1, k_2, \ldots, k_N) = \text{cum}_{\gamma,n}(k_1, k_2, \ldots, k_N). (33)
\]

The integrand on the right-hand side of the second inequality above can now be upper bounded by
\[
2M^{(q+1)/2} \mathfrak{H}_n(\omega) \mathfrak{H}_n^{(\gamma)}(\omega) \mathfrak{H}_n(\omega) \mathfrak{H}_n^{(\gamma)}(\omega)
\]
where we use (23) and (24) and take into account that \(\mathfrak{H}_n(\omega)\) and \(\mathfrak{H}_n^{(\gamma)}(\omega)\) are less than or equal to one. The upper bound given by (35) is independent of \(r\) and integrable; its integral equals \(2M^{(q+1)/2}(2\pi)^q\). By taking \(\gamma \) into account, we derive from Lebegue’s dominated convergence theorem that the upper bound in (34) tends to zero when \(\gamma \) tends to \(+\infty\).

**Corollary 2:** With the same assumptions as those of Theorems 1 and 4, let \(\mathcal{P} = (\mathcal{S}_X, \{\mathcal{S}^{(\gamma)}_{X,n}\}, \{q\}^n\) be a path of the Shannon M-DWPT tree of the PSW process \(\mathcal{S}_X\), the decomposition being achieved by using filters \(h_{m,n}^{(\gamma)}(\omega)\), \(m = 0, 1, \ldots, M - 1\). Let \(\text{cum}_{\gamma,n}(k_1, k_2, \ldots, k_N)\) stand for the discrete random process returned at node \((j, n)\) by the projection of X on \(\mathcal{S}^{(\gamma)}_{X,n}\).

Then, when \(j \to \infty\), the sequence \(\text{cum}_{\gamma,n}(k_1, k_2, \ldots, k_N)\) converges in the following distributional sense to a Gaussian process with variance \(\gamma(\omega,r)\): for every \(x \in \mathbb{R}^q\) and every \(\epsilon > 0\), there exists a natural number \(j_0 = j_0(x, \epsilon)\) such that, for every natural number \(j \geq j_0\), the absolute value of the difference between the value at \(x\) of the probability distribution of the random vector
\[
\left( \frac{\text{cum}_{\gamma,n}(k_1)}{\text{cum}_{\gamma,n}(k_2)}, \ldots, \frac{\text{cum}_{\gamma,n}(k_d)}{\text{cum}_{\gamma,n}(k_1)} \right)
\]
and the value at \(x\) of the \(q\)-variate normal distribution \(N(0, \gamma(\omega,r)I_q)\) is less than \(\epsilon\).

**Proof:** A straightforward consequence of Theorems 1 and 3.

The following result describes the asymptotic behavior of the cumulant of the discrete random process returned at node \((j, n)\) in the case of practical interest where the M-DWPT of the PSW process \(\mathcal{S}_X\) is achieved via decomposition filters satisfying (21).

**Theorem 4:** Let X be a zero-mean second-order strictly stationary random process, continuous in quadratic mean. Assume that the polyspectrum \(\gamma_0\) of X is an element of \(L^\infty(\mathbb{R}^q)\) for every natural number \(q \geq 1\) and that the spectrum \(\gamma_0\) of X is supported within \([-\pi, \pi]\). Consider the Shannon M-DWPT of the PSW process \(\mathcal{S}_X\) when the decomposition filters \(h_{m,n}^{(\gamma)}(\omega)\), \(m = 0, 1, \ldots, M - 1\), satisfy (21).

Let \(\text{cum}_{\gamma,n}(k_1, k_2, \ldots, k_N)\) stand for the cumulative of order \(q + 1\) of the discrete random process resulting from the projection of X on \(\mathcal{S}^{(\gamma)}_{X,n}\). We have,
\[
\lim_{n \to +\infty} \text{cum}_{\gamma,n}(k_1, k_2, \ldots, k_N) = \text{cum}_{\gamma,n}(k_1, k_2, \ldots, k_N).
\]

**References**


Acknowledgment

The authors are very grateful to the reviewers for their insightful comments, in particular, their suggestions concerning the presentation and the overall organization of this paper.
Optimized Analog Flat Filter Design

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Abstract—This paper proposes a systematic approach for the optimized design of analog filters, which includes all well-known classical analog filters as a special case. All specifications including the conventional ones and also filter flatness degrees are explicitly incorporated into design process. Several numerical examples are presented to demonstrate the efficiency and flexibility of the proposed method.

Index Terms—Analog filter, convex analysis, global optimization, semi-definite programming (SDP).

I. INTRODUCTION

Analog filters are indispensable parts in interface with the analog real world. Analog and digital circuits are often implemented together in the real world. Analog and digital circuits are often implemented together on the same integrated circuit chip [1], [3], [4], [12], [13]. Also, the most popular approach in digital infinite-impulse-response (IIR) filters design is based on transformation methods from analog counterpart [2], [9], [10]. There is no doubt that design of analog filters is a fundamental problem in signal processing, communications, and control.

In contrast to digital IIR filter design, the classical design of analog IIR filters looks rather complete, so there is not much further development. The Chebyshev min-max approximation works so well. Each classical filter is optimal in some sense. Given the filter order, the stopband ripple and the cut-off frequency the Chebyshev filter has the least peak ripple in the passband among all pole filters [9]. On the other hand, given the filter order, the passband ripple and the cut-off frequency the inverse Chebyshev filter has the least peak ripple in the stopband among the maximally flat (at passband) filters [9]. Finally, given three out of four design parameters: the filter order, the passband ripple, the transition band width, the stopband ripple, the elliptic (Cauer) filter minimizes the only one remaining design parameter. In terms of these four design parameters, the optimized design is thus complete.

However, because of the nature of the minimax optimality, all classical filters are unable to satisfy additional regularity or flatness conditions, which are desirable in many practical applications. For instance, Chebyshev filters are maximally flat at the stopband (like other all-pole filters) but cannot be flat for any degree at the passband. The inverse Chebyshev filters are maximally flat (at passband) but cannot be flat for any degree at the stopband. Elliptic (Cauer) filters cannot be flat for any degree at the both passband and stopband. In our previous work [6], an alternative design to Chebyshev and inverse Chebyshev filters has been proposed. The designed filters have the same structure as of Chebyshev and inverse Chebyshev ones but they possess additional flatness for any degree at either passband or stopband. The design is based on a new semi-definite programming (SDP) formulation, which also includes Chebyshev and inverse Chebyshev filter designs as a special case. This paper is a further development of [6], where a complete formulation for problem of designing general flat filters is proposed and tested in several examples.

The rest of the paper is structured as follows. Section II describes mathematical tools that will be used throughout the paper. Section III presents the reduced order SDP formulation for generalized elliptic filters. In Section IV, various numerical examples are given to demonstrate the viability of the proposed method. Finally, concluding remarks are presented in Section V.

The following notation is used in the paper. Vectors and matrices will be represented by italicized bold lower case and upper case letters, respectively. The superscript \(^T\) denotes the transpose (without conjugation) whereas the superscript \(^*\) denotes Hermitian transpose. Symbols \(\mathbb{R}\) and \(\mathbb{C}\) are used to denote real and complex spaces. Real part and imaginary part of a complex number \(w\) are denoted by \(\Re(w)\) and \(\Im(w)\), respectively. The round-down and the round-up operations to the closest integers of a number \(a\) are respectively denoted by \([a]\) and \([a]\). The standard notation \(X \geq 0\) defines a positive semi-definite Hermitian matrix, while \(\langle X, Y \rangle = \text{Trace}(X^*Y)\). \(\alpha\) and \(I_k\) are the zero and identity matrices of size \(k \times k\), respectively. The notation \(\text{diag}(A, B)\) means the matrix

\[
\begin{bmatrix}
A & 0 \\
0 & B
\end{bmatrix}
\]

For a given set \(C \subseteq \mathbb{R}^n\) its convex hull (conic hull), denoted by \(\text{conv}(C)\) (cone \(C\)), is the smallest convex set (cone) in \(\mathbb{R}^n\) that contains \(C\).

II. OPTIMIZATION TOOL: SDP

For \(\varphi_a(\omega) = (1, \omega, \omega^2, \ldots, \omega^{2N-1})^T\), a polynomial curve \(C_{a,b} \subseteq \mathbb{R}^{2N+1}\) is defined as \(C_{a,b} := \{ \varphi_a(\omega) : \omega \in [a, b]\} \subseteq \mathbb{R}^{2N+1}\), and its polar \(C^*_{a,b}\) is given by \(C^*_{a,b} = \{ u \in \mathbb{R}^{2N+1} : \langle u, v \rangle \geq 0 \ \forall v \in C_{a,b}\}\). For an integer \(k\) define the linear matrix valued functions

\[
T_k(y) = \begin{bmatrix}
y_0 & y_1 & \ldots & y_k \\
y_1 & y_2 & \ldots & y_{k+1} \\
\vdots & \vdots & \ddots & \vdots \\
y_k & y_{k+1} & \ldots & y_{2k}
\end{bmatrix}
\]